Chapter 7

Stability, Error Estimate and Limiters of Discontinuous Galerkin Methods

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Chapter Outline

ABSTRACT

In this chapter, we review the development of discontinuous Galerkin (RKDG) methods for conservation laws and focus on the stability, error estimates and limiters for RKDG methods. The stability and error estimates are core properties of RKDG, and limiter is an important component of RKDG methods for solving conservation laws with strong shocks in the solutions, which is applied to detect discontinuities and control spurious oscillations near such discontinuities.

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1 INTRODUCTION

In this chapter, we review some development of discontinuous Galerkin (DG) method and focus mainly on the stability, error estimate and limiters for the d-dimensional conservation law

$$\partial_t u + \nabla \cdot f(u) = 0, \tag{1}$$

with the initial solution $u(x, 0) = u_0(x)$, where $f(u)$ is the given smooth flux function. For simplicity, we mainly consider the case when the exact solution is periodic or compactly supported.

After the first version of DG method, which was introduced by Reed and Hill (1973), in the framework of neutron linear transport, the DG method has been paid more and more attention, because of its many advantages. For example, this method has strong stability and optimal accuracy to capture discontinuous jump sharply and combines the advantages of finite element method and finite difference method. An important development in the DG method is in the late 1980s, when Cockburn and Shu (1991, 1989, 1998) and Cockburn et al. (1989, 1990) combine the Runge–Kutta time discretization and the DG spatial discretization, with exact or approximate Riemann solvers as interface fluxes and total variation bounded (TVB) limiter (Shu, 1987) to achieve nonoscillatory properties for strong shocks, to easily solve *nonlinear* time-dependent hyperbolic conservation laws (1). These schemes are termed Runge–Kutta discontinuous Galerkin (RKDG) methods. Till now there have been many published papers in this field, for example, see the review paper and books (Cockburn, 1998; Cockburn and Shu, 2001) and the others.

This chapter shall focus on two issues. One is the theoretical analysis for this kind of method, and the other is the design and application of limiters.

The error estimates of DG method have been paid attention to by many authors. Lasaint and Raviart (1974) proved first the suboptimal order for general triangulations, and optimal order for Cartesian grids. Later, Johnson and Pitkäranta (1986) proved a quasi-optimal rate of convergence for general triangulation and Peterson (1991) confirmed its sharpness. Note that optimal error estimate can be achieved for some meshes with special structure (Richter, 1988; Cockburn et al., 2008). However, the above discussions are carried out for either the steady problem or the space-time DG method and semidiscrete DG method for unsteady problems. In this chapter, we mainly discuss the error estimates to the fully discrete RKDG methods for the conservation law (1), whether the exact solution has sufficient smoothness or not.

One of the main difficulties in using RKDG methods to solve (1) with possibly strong shocks or contact discontinuities is that the numerical solution might cause spurious oscillations. These spurious oscillations might lead to nonlinear instability. One common strategy to control these oscillations is to apply nonlinear limiters to RKDG methods. Many limiters have been studied in the literature for RKDG methods, such as the *minmod*-type TVB limiter (Cockburn and Shu, 1989, 1998; Cockburn et al., 1989, 1990), the moment-based limiter (Biswas et al., 1994) and an improved moment limiter (Burbeau et al., 2001). These limiters belong to the slope-type limiters and they do control oscillations very well at the price of possibly degrading the accuracy of the numerical solution at smooth extrema. Another type of limiters is the WENO-type limiters, which are based on the weighted essentially nonoscillatory (WENO) methodology (Jiang and Shu, 1996; Liu et al., 1994) and can achieve both high-order accuracy and nonoscillatory property near discontinuities. This type of limiters includes the WENO limiter (Qiu and Shu, 2005c; Zhu et al., 2008) and the HWENO limiter (Luo et al., 2007; Qiu and Shu, 2004, 2005b), which use the classical WENO finite volume methodology for reconstruction and thus require a wide stencil, especially for higher order methods. More recently, the new WENO limiting procedures were developed for RKDG methods (Zhong and Shu, 2013; Zhu et al., 2013, 2016, submitted for publication) on both the structure and unstructured meshes, and the idea is to reconstruct the entire polynomial on the target cell by a convex combination of polynomials on this cell and its immediate neighbouring cells, with suitable adjustments for conservation and with the nonlinear weights of the convex combination following the classical WENO procedure. The subcell limiting procedures were developed in Dumbser et al. (2014) and Zanotti et al. (2015a,b), which breaks the DG cell into subcells and then uses WENO ideas for limiting.

The chapter is organized as follows. In Section 2, we present the framework of RKDG method. Then we present some stability results in Section 3, and some error estimates in Section 4. In Section 5, we introduce some good limiter used in the RKDG method. Finally, some concluding remarks are given in Section 6.

2 IMPLEMENTATION OF DG METHODS

We start with the description in the bounded interval $I = (0, 1)$; however, it works also in multi-dimensions. Divide it into N cells with boundary points $0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1$ and denote the cell size of $I_i = [x_{i-1/2}, x_{i+1/2}]$ by $h_i = x_{i+1/2} - x_{i-1/2}$. The maximum cell size is denoted by $h = \max_i h_i$. For simplicity of presentation, we would like to assume that the used mesh is quasi-uniform; namely, there exists a positive constant C independent of h, such that $Ch_i \geq h$ for every $i = 1, 2, \ldots, N$, as h goes to zero.

The numerical solution and the test function are both considered in the following discontinuous finite element space:

$$V_h = V_h^k = \{ v \in L^2(I) : v|_{I_i} \in \mathbb{P}^k(I_i), i = 1, \ldots, N \}, \tag{2}$$

where $\mathbb{P}^k(I_i)$ is the space of polynomials of degree at most $k \geq 0$ on the cell I_i. Note that the functions in V_h are allowed to have discontinuities across element interfaces. For any function $v_h \in V_h$, there are two limits along different directions at each element boundary point, namely, the left-value v_h^- and the right-value v_h^+. Further, the jump and mean, respectively, are denoted by

$$[\![v_h]\!] = v_h^+ - v_h^- \quad \text{and} \quad \{\!\{v_h\}\!\} = \frac{1}{2}(v_h^+ + v_h^-). \tag{3}$$

2.1 Semidiscrete Version

First we define the semidiscrete DG(k) method as follows. We would like to find the numerical solution $u_h(t) \in V_h$ for any time $t > 0$, such that

$$(u_{h,t}, v_h) = \mathcal{H}(u_h, v_h), \quad \forall\, v_h \in V_h, \tag{4}$$

where the global DG spatial discretization is defined by

$$\mathcal{H}(u_h, v_h) = \sum_{1 \leq i \leq N} \left[\hat{f}(u_h)_{i+1/2} [\![v_h]\!]_{i+1/2} + \int_{I_i} f(u_h) v_{h,x} \mathrm{d}x \right]. \tag{5}$$

Here $(\cdot, \cdot)$ is the usual inner product in $L^2(I)$, and

$$\hat{f}(u_h) = \hat{f}(u_h^-, u_h^+) \tag{6}$$

is the numerical flux defined on each element boundary point. For example, the Lax–Friedrich numerical flux

$$\hat{f}(u_h^-, u_h^+) = \frac{1}{2}\left[f(u_h^-) + f(u_h^+) - C [\![u_h]\!] \right] \tag{7}$$

is used widely in practice, where $C = \max |f'(u)|$. Obviously, it is an E-flux or monotone flux, since it is not decreasing for the first argument and not increasing for the second argument. More numerical flux can be found in Qiu et al. (2006).

The initial solution is usually given as the approximation of the given solution $u_0(x)$. For example, $u_h^0 = \pi_h u_0(x)$ is the local L^2-projection of $u_0(x)$, such that

$$\int_0^1 (\pi_h u_0(x) - u_0(x)) v_h(x) \mathrm{d}x = 0, \quad \forall v_h \in V_h. \tag{8}$$

Below we would not mention the setting of initial solution, since it only affects the numerical error, but not the numerical stability.

Remark 1. Eq. (5) is obtained by simple summary of DG formulation on each element, as the periodic boundary condition has been used in the above process. The other boundary condition, for example, the inflow boundary condition (Zhang, 2011), can be treated with in a similar way.

2.2 SSPRK Algorithms

The second feature of RKDG method is the time marching. One of the famous and successful treatment is adopting the explicit total variation diminishing Runge–Kutta time marching; please refer to the series papers of Cockburn and Shu (1991, 1989, 1998) and Cockburn et al. (1989, 1990). Now this kind of time marching has been considered in the term *strong stability-preserving* (SSP) algorithms (Gottlieb et al., 2001).

It is to say that we would like to seek the solution u_h^n at the time level $t^n = n\tau$ step by step, where τ is the time step. The time step could actually change from step to step; for simplicity, in this chapter we take it as a constant. According to the Osher–Shu representation (Shu and Osher, 1988), the general construction of RKDG(s, r, k) method is given as follows, where s and r are the stages and the order of the used Runge–Kutta time marching, respectively, and k is the degree of piecewise polynomials. Assume that the numerical solution $u_h^n \in V_h$ has been obtained, we will solve successively each stage solution $u_h^{n,\ell+1} \in V_h$ for $\ell = 0, 1, 2, \ldots, s-1$, by virtue of the variation form

$$(u_h^{n,\ell+1}, v_h) = \sum_{0\le\kappa\le\ell} a_{\ell\kappa}(u_h^{n,\kappa}, v_h) + b_\ell \mathcal{H}(u_h^{n,\ell}, v_h)\tau, \quad \forall v_h \in V_h, \tag{9}$$

where $u_h^{n,0} = u_h^n$ and $u_h^{n+1} = u_h^{n,s}$. Note that the coefficients, $a_{\ell\kappa}$ and b_ℓ, are given for the used time marching, with the distinguish property that they are all nonnegative. The parameters in the RKDG(3, 3, k) and RKDG(2, 2, k) are given in Table 1.

The above fully discrete schemes are actually implemented explicitly because the mass matrix is easy to be inverted due to the block diagonal structure. The mass matrix will be diagonal when a local orthogonal basis is chosen for polynomials on each element.

TABLE 1 Parameters: Left: RKDG(2,2,k); Right: RKDG(3,3,k)

			1			1
1		1	$\frac{3}{4}$	$\frac{1}{4}$		$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$

2.3 Limiters

The method described in Sections 2.1 and 2.2 can compute solutions to (1) which are either smooth or have weak shocks and other discontinuities without further modification. If the discontinuities are strong, however, the scheme will generate significant oscillations and even nonlinear instability. To avoid such difficulties, a nonlinear limiter procedure is used after each Runge–Kutta inner stage (or after the complete Runge–Kutta time step) to control the numerical solution. There are many limiters which exist in the literature, for example, the minmod-type limiters, the moment-based limiters, the improved moment-based limiters and the WENO-type limiter. We will describe these limiters in details in Section 5.

3 STABILITY

In this section, we collect some stability results on the DG methods. To show this, let us recall some inverse properties of finite element space V_h. Specially, for any function $v_h \in V_h$, there hold[a]

$$\| (v_h)_x \| \le \mu_1 h^{-1} \| v_h \|, \quad \| v_h \|_{\Gamma_h,\pm} \le \mu_2 h^{-1/2} \| v_h \|, \quad \| v_h \|_{\Gamma_h,*} \le \mu_3 h^{-1/2} \| v_h \|, \tag{10}$$

where the inverse constants, μ_1, μ_2 and μ_3, are independent of v_h and solely depend on the degree k of the piecewise polynomials (Zhang and Shu, 2009, in preparation). The sharp values are listed in Table 2 for $k \le 4$. Here Γ_h denotes all element boundary points,

$$\| v \|_{\Gamma_h,\pm} = \left[\sum_{i=1}^{N} |v_{i+1/2}^{\pm}|^2 \right]^{1/2} \quad \text{and} \quad \| v \|_{\Gamma_h,*} = \left[\| v \|_{\Gamma_h,-}^2 + \| v \|_{\Gamma_h,+}^2 \right]^{1/2}. \tag{11}$$

TABLE 2 Inverse Constants on the Uniform Mesh (Zhang and Shu, 2009, in preparation): $k \le 4$

k	0	1	2	3	4
μ_1	0	$\sqrt{12} \approx 3.46$	$\sqrt{60} \approx 7.75$	$\sqrt{2(45+\sqrt{1605})} \approx 13.04$	$\sqrt{2(105+3\sqrt{805})} \approx 19.50$
μ_2	1	2	3	4	5
μ_3	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{12}$	$\sqrt{20}$	$\sqrt{30}$

[a]Note that the above L^2-norm of derivative should be understood element by element.

For the linear case, the stability in L^2-norm is clear, for both semidiscrete version and fully discrete version. However, for the nonlinear case, the answer is not completely clear till now.

3.1 Linear Stability in L^2-Norm

Assume $f(u) = \beta u$ with β being a given constant. In this case, $\hat{f}(u_h) \equiv \hat{f}(u_h^-, u_h^+)$ is the so-called upwind numerical flux

$$\hat{f}(u_h) = \hat{f}(u_h^-, u_h^+) = \begin{cases} \beta u_h^-, & \text{if } \beta > 0, \\ \beta u_h^+, & \text{if } \beta < 0; \end{cases} \tag{12}$$

thus the global DG spatial discretization in (4) is defined explicitly by

$$\mathcal{H}(u_h, v_h) = \sum_{1 \le i \le N} \left[\hat{f}(u_h)_{i+1/2} [\![v_h]\!]_{i+1/2} + \int_{I_i} \beta u_h v_{h,x} \mathrm{d}x \right]. \tag{13}$$

The following three properties (Zhang and Shu, 2009, 2010) about the DG space discretization provide an important distribution to the success of DG method, which can achieved after a simple application of integration by parts, the inverse properties together with Cauchy–Schwarz inequality.

Lemma 1. *The bilinear functional has the approximating antisymmetric property*

$$\mathcal{H}(\psi, \phi) + \mathcal{H}(\phi, \psi) = -\sum_{1 \le j \le N} |\beta| [\![\phi]\!]_{j+\frac{1}{2}} \cdot [\![\psi]\!]_{j+\frac{1}{2}}, \quad \forall \phi, \psi \in V_h. \tag{14}$$

As a direct conclusion, the bilinear functional has the negative semidefined property

$$\mathcal{H}(\phi, \phi) = -\frac{1}{2} \sum_{1 \le j \le N} |\beta| [\![\phi]\!]_{j+1/2}^2 = -\frac{1}{2} |\beta| \, \| \, [\![\phi]\!] \, \|_\Gamma^2, \quad \forall \phi \in V_h. \tag{15}$$

Furthermore, the bilinear functional is continuous and bounded in $V_h \times V_h$, in the sense

$$|\mathcal{H}(\phi, \psi)| \le |\beta| \, \| \psi \| \left[\mu_1 h^{-1} \| \phi \| + \mu_2 h^{-1/2} \| [\![\phi]\!] \|_\Gamma \right], \quad \forall \psi, \phi \in V_h. \tag{16}$$

Now the L^2-norm stability of semidiscrete version is easy. It is followed from the negative semidefined property that

$$\frac{1}{2} \| u_h(T) \|^2 + \frac{|\beta|}{2} \int_0^T \| \, [\![u_h(t)]\!] \, \|_\Gamma^2 \, \mathrm{d}t \le \frac{1}{2} \| u_h(0) \|^2, \tag{17}$$

which reflects the subtle built-in dissipation mechanism of the DG method and allows more accurate than the standard Galerkin methods. However, this stability mechanism from the square of jumps is very weak; hence the time marching must be treated carefully, if the time step only satisfies the standard

CFL condition that the ratio of the time step over the mesh's size is fixed in a constant. Although RKDG(r, s, k) method has been used successfully in numerical practice, the stability analysis is nontrivial under the SSP framework (Gottlieb et al., 2001), because Euler-forward time marching of DG method is linearly unstable under the standard CFL (Courant–Friedrichs–Lewy) condition. Thus we have to find another way to prove the L^2-norm stability in theory.

Roughly speaking, there are two main strategies to do this. The trivial analysis is the so-called Fourier's technique, which can give the sharp CFL condition or the maximum CFL number. But, this technique demands too much assumptions that the used mesh is uniform and the boundary condition is given in periodical status. It is also hard to develop into the linear varying-coefficient problems, the nonlinear problems, the general boundary condition and multidimensional problems. The second strategy is energy analysis to overcome the above difficulties. This motivation comes from the optimal error estimate for two RKDG methods to solve the nonlinear conservation law (Zhang and Shu, 2004, 2010), which is obtained by virtue of the suitable projection and the stability analysis for the linear case.

Restricted by the page limitation, we would like to focus on the energy analysis and present only the sketch. The following material is partially taken from Zhang and Shu (2009). The important idea is to introduce some differences of stage solutions

$$\mathbb{D}_\ell u_h^n = \sum_{0\le\kappa\le\ell} \sigma_{\ell\kappa} u_h^{n,\kappa}, \quad \ell = 1,\ldots,s, \tag{18}$$

such that

$$(\mathbb{D}_{\ell+1} u_h^n, v_h) = \frac{\tau}{\ell+1}\mathcal{H}(\mathbb{D}_\ell u_h^n, v_h), \quad \forall\, v_h \in V_h, \tag{19}$$

holds for $\ell = 0, 1, \ldots, s-1$. Here and after we denote $\mathbb{D}_0 u_h^n = u_h^n$ for simplicity. The combination coefficients $\gamma_{\ell\kappa}$ are given constants independent on the numerical solution and satisfy the consistent condition $\sum_{0\le\kappa\le\ell}\sigma_{\ell\kappa} = 0$. This purpose is easily achieved by a series of linear combinations of (9). It depends on the used time marching. For example, in RKDG(2,2,k) they are defined as

$$\mathbb{D}_1 u_h^n = u_h^{n,1} - u_h^n, \quad \mathbb{D}_2 u_h^n = u_h^{n+1} - u_h^{n,1}, \tag{20}$$

and in RKDG(3,3,k) they are defined as

$$\mathbb{D}_1 u_h^n = u_h^{n,1} - u_h^n, \quad \mathbb{D}_2 = 2u_h^{n,2} - u_h^{n,1} - u_h^n, \quad \mathbb{D}_2 = u_h^{n+1} - 2u_h^{n,2} + u_h^n. \tag{21}$$

These differences of stage solutions can be looked upon as the approximation of different order time derivatives, due to (19) and the following observation

$$\| \mathbb{D}_{\ell+1} u_h^n \| \le \frac{1}{\ell+1} K\lambda \| \mathbb{D}_\ell u_h^n \|, \quad \ell = 0,1,\ldots,s-1, \tag{22}$$

where K is a bounding constant depending solely on the inverse constants.

The next procedure is to establish the energy equation, such as

$$\| u_h^{n+1} \|^2 - \| u_h^n \|^2 = \sum_{0 \le \ell \le s-1} \gamma_\ell \mathcal{H}(u_h^{n,\ell}, u_h^{n,\ell}) \tau + \sum_{1 \le i \le j \le s} \omega_{ij} (\mathbb{D}_i u_h^n, \mathbb{D}_j u_h^n), \tag{23}$$

where γ_ℓ and ω_{ij} are some given constants and can be sought by a trivial algebraic manipulation. Assume furthermore

$$\gamma_\ell \ge 0, \quad \ell = 0, 1, \ldots, s-1, \tag{24}$$

which can be verified at least for both RKDG(2,2,k) and RKDG(3,3,k) methods. Two terms on the right-hand side of (23) have distinguishing meanings. The first one represents the elemental stability owing to the DG spatial discretization, and the second one represents the interactional function owing to the time marching.

The main and difficult work in the whole energy analysis is how to control sharply the second term on the right-hand side of (23). This work depends strongly on the properties of DG spatial discretization, as well as the deep investigation of different stability mechanisms. For example, an additional numerical stability in RKDG(3,3,k) method shows up explicitly in the term $- \| \mathbb{D}_2^n \|^2$. It is to say that the dissipative nature of the RKDG(3,3,k) method comes from not only DG space discretization but also time marching. However, the stability mechanisms in RKDG(2,2,k) method are totally different. To obtain strong stability in L^2-norm, we have to balance the dissipation of DG spatial discretization and the antidissipation of the used time marching, since the used time marching does not belong to the so-called $A(\pi)$-stable algorithm in the terminology of the ODE solvers. This aim to obtain stability in L^2-norm can be completed by using a special property holding only for $k = 1$, namely,

$$\| \mathbb{D}_2 u_h^n \| \le \frac{1}{2} \mu_2 (|\beta| \lambda \tau)^{1/2} \left[\| [\![\mathbb{D}_1 u_h^n]\!] \|_\Gamma + \lambda \mu_1 \| [\![\mathbb{D}_0 u_h^n]\!] \|_\Gamma \right]. \tag{25}$$

This inequality can be obtained by filtering the average of numerical solution in each element, which can be extended to the generalized slope function for high-order piecewise polynomials (Cheng and Shu, 2010). For more detailed analysis, see Zhang and Shu (2009).

Finally, the strong L^2-norm stability results for two types of RKDG method can be established under the standard CFL condition.

Theorem 1. *Let* $\mathrm{u_h}$ *be the numerical solution of the RKDG(s, r, k) scheme. If the CFL number* $\lambda = |\beta| \tau h^{-1}$ *satisfies*

$$\mu_2^2 \lambda (\sqrt{2} + \mu_1 \lambda)^2 \le 2, \tag{26}$$

for $\mathrm{s} = \mathrm{r} = 2$ *and* $k \le 1$, *or satisfies*

$$\lambda \le \frac{12}{3\mu_3^2 + \sqrt{9\mu_3^4 + 48(\mu_1 + \sqrt{2}\mu_2 \mu_3)^2}}, \tag{27}$$

for $\mathrm{s} = \mathrm{r} = 3$ *and arbitrary* $\mathrm{k} \ge 0$, *then* $\| u_h^{n+1} \| \le \| u_h^n \|$ *for any* n.

Remark 2. For the uniform mesh, it follows from the inverse constant listed in Table 2 and Theorem 1 that the maximum of CFL number for RKDG(2,2,1) method and RKDG(3,3,2) method, respectively, is approximated to be 0.1391 and 0.0904. There exists a gap to the sharp CFL number, 0.33 and 0.20 for above two schemes, respectively, which has been given by the Fourier technique and the numerical experiments.

Remark 3. Along the same line, we are able to use energy technique and obtain L^2-norm stability of other RKDG methods (Zhang and Shu, in preparation) with either second-order or third-order SSPRK time marching, for example, RKDG(3,2,1), RKDG(4,2,1), and RKDG(4,3,k) methods. However, how to extend the above work to the RKDG method with higher order Runge–Kutta time marching is an open problem till now.

3.2 Nonlinear Stability

There are a few stability results for the nonlinear case. The L^2-norm stability of semidiscrete DG method has been proved for arbitrary degree $k \geq 0$ by Jiang and Shu (1994), due to the famous *cell entropy inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} \| u_h \|_{I_i}^2 + \hat{F}_{i+\frac{1}{2}} - \hat{F}_{i-\frac{1}{2}} \leq 0, \tag{28}$$

where

$$\hat{F} = \hat{f}(u_h^-, u_h^+) u_h^- - \int_0^{u_h^-} f(s) \mathrm{d}s$$

is consistent with the entropy flux for the square entropy. The above cell entropy inequality can be proved for semidiscrete and Euler-forward (thus SSP-type Runge–Kutta) time marching, if the piecewise constant is considered. However, the nonlinear stability for the fully discrete RKDG method is an open problem till now, for high-order piecewise polynomials.

Given the function of limiter in RKDG methods, there are some more results in stability. For example, it has been proved in Cockburn (1998) that the numerical solution of RKDG method satisfies the total variation diminishing of means property

$$|\bar{u}_h^n|_{TV(0,1)} \leq |\bar{u}_h^0|_{TV(0,1)}, \quad \forall n \geq 0, \tag{29}$$

under a suitable CFL condition, when some slope limiter (such as MUSCL limiter) in each stage updating is used. Here $\bar{u}_h^n$ is the piecewise constant, made up of the average of u_h in each element. Similarly, the TVBM properties have been proved for some limiters, if the limiter can overcome the order reduction on the so-called sonic point. The detailed contents about limiter's implementation will be given in Section 5.

4 ERROR ESTIMATES

To show the numerical advantage of RKDG method, the error estimate is necessary. In this section, we mainly present several optimal (and/or quasi-optimal) error estimates in L^2-norm and so on.

4.1 Scalar Equation with Smooth Solution

We start from the scalar conservation law with sufficiently smooth solution. The material presented in this section is mainly taken from Zhang and Shu (2004, 2010). They introduce a quantity at each element boundary point

$$\alpha(\hat{f};w_h)=\begin{cases}\left\{f(\{\!\{w_h\}\!\})-\hat{f}(w_h^-,w_h^+)\right\}/[\![w_h]\!], & \text{if } [\![w_h]\!]\neq 0,\\ \dfrac{1}{2}f'(\{\!\{w_h\}\!\}), & \text{if } [\![w_h]\!]=0,\end{cases} \tag{30}$$

to describe the numerical viscosity coming from the numerical flux $\hat{f}$. It is easy to see that $\alpha(\hat{f};w_h)=|\beta|/2$ for the upwind flux (12). If $\hat{f}$ is an E-flux, a trivial analysis shows the following properties:

- Obviously there holds $\alpha(\hat{f};w_h)\geq 0$. Therefore, the numerical stability of semidiscrete DG method can be expressed approximately in an explicit form $\alpha(\hat{f};u_h)[\![u_h]\!]^2$ at every element endpoints. Note that we do not assume $\alpha(\hat{f};w_h)$ to have a uniform lower-bounded above zero, which leads to a major difficulty in analysis.
- It follows from the above definition that

$$\frac{1}{2}|f'(\{\!\{w_h\}\!\})|\leq\alpha(\hat{f};w_h)+C_*|[\![w_h]\!]|, \tag{31}$$

where C_* is a bounding constant depending solely on the nonlinearity of given flux $f(\cdot)$. Furthermore, it is reasonable to assume that $\alpha(\hat{f};w_h)$ is Lipschitz continuous with each argument. Hence, we can see that $\alpha(\hat{f};w_h)\approx\dfrac{1}{2}|f'(\{\!\{w_h\}\!\})|$ in first order of jump.

Due to the explicit expression on the numerical viscosity, we can obtain the optimal error estimate for semidiscrete version (Xu and Shu, 2007) and fully discrete version (Zhang and Shu, 2004, 2010). The next theorem shows the optimal error estimates for the fully discrete version.

Theorem 2. *Assume $T=M\tau$ for simplicity. For the RKDG(s, r, k) method, there holds the following optimal (or quasi-optimal) error estimate:*

$$\|u_h^M-u(\cdot,T)\|\leq C(h^{k+\sigma}+\tau^r), \tag{32}$$

under the standard CFL condition $\tau\leq\gamma h$, where γ is a suitable CFL number. Here $s=r=2$ and $k=1$ for RKDG(2,2,1) method, and $s=r=3$ and arbitrary $k\geq 1$ for RKDG(3,3,k) method. In general, $\sigma=1/2$ when the E *-flux is used, and moreover $\sigma=1$ when the upwind numerical flux is used.*

The main tools used in analysis are the projection's techniques and the detailed investigation on the numerical flux. To deal with those errors resulting from the time marching, those discussion and techniques used in the stability analysis are important and useful, too. For more details, see Zhang and Shu (2004, 2010).

4.2 Symmetrizable System with Smooth Solution

The above error estimate can be extended to the symmetrizable system (1), including Euler equations. Namely, there exists a mapping $u(v):\mathbb{R}^m \to \mathbb{R}^m$ so that when transformed into

$$u_v v_t + f_v v_x = 0,$$

the matrix $\mathbb{Q}(v) \equiv u_v$ is symmetric positive definite matrix and $f_v = f_u u_v$ is also symmetric matrix. The main development is the abstract definition of generalized E-flux and how to describe the corresponding numerical viscosity. The following material is taken from Luo et al. (2015).

Definition 1. The numerical flux $\hat{f}(w_h^-, w_h^+)$ is local Lipschitz continuous with each argument and consistent with $f(w)$. It is called a generalized E-flux, if there exist a rotation position s_κ and an adjusting matrix $\mathbb{Y}_\kappa$, such that

$$[\![w_h]\!]^\top \mathbb{Q}(s_\kappa)\{f(r_\kappa) - \hat{f}(w_h^-, w_h^+)\} + [\![w_h]\!]^\top \mathbb{Y}_\kappa [\![w_h]\!] \geq 0, \quad \kappa = 1,2, \tag{33}$$

for both $r_1 = w_h^-$ and $r_2 = w_h^+$, where $s_\kappa = s_\kappa(w_h^-, w_h^+)$ lies in the standard super-rectangle with two vertices w_h^+ and w_h^-, and each element in $\mathbb{Y}_\kappa = \mathbb{Y}_\kappa(w_h^-, w_h^+)$ has first order of jump, with the common bound depending only on the local Lipschitz constant of $\mathbb{Q}(\cdot)$ in the above super-rectangle.

This definition covers many numerical fluxes for the system, for example, Lax–Friedrich flux. Now we can introduce the numerical viscosity matrix at each element boundary point

$$\mathcal{A}(\hat{f}; w_h) \equiv \frac{1}{2}\mathbb{Q}(s_1)\mathcal{D}\hat{f}^{(1)}[w_h^-, w_h^+] - \frac{1}{2}\mathbb{Q}(s_2)\mathcal{D}\hat{f}^{(2)}[w_h^-, w_h^+], \tag{34}$$

where $\hat{f}^{(1)}(r) = \hat{f}(r, w_h^+)$ and $\hat{f}^{(2)}(r) = \hat{f}(w_h^-, r)$ are single-value functions with respect to the vectored-variable r. Here we have used the concept of the generalized Newton difference quotient $\mathcal{D}g[a,b]$, for any given function $g = (g_1, g_2, \ldots, g_m)^\top : \mathbb{R}^m \to \mathbb{R}^m$ between two points a and b in m-dimensional space. Specifically, each element in $\mathcal{D}g[a,b]$ is defined by

$$(\mathcal{D}g[a,b])_{ij} = \frac{g_i(a^{(j-1)}) - g_i(a^{(j)})}{a_j - b_j}, \tag{35}$$

where $a^{(j-1)}$ is an m-dimensional vector defined as

$$a^{(j)} = (b_1, \ldots, b_{j-1}, b_j, a_{j+1}, \ldots, a_m)^\top, \quad j = 1, 2, \ldots, m-1, \tag{36}$$

together with $a^{(0)} = a$ and $a^{(m)} = b$. If the denominator is equal to zero, the term on the right-hand side of (35) should be understood as the limit when the denominator goes to zero.

Now we can have almost the same error estimate as Theorem 2, by using energy technique with more careful treatment on the rotation in the middle point and boundary point of each element. The discussion is more complex than scalar case, because the numerical viscosity matrix (34) is not less than zero in an approximation status. For more details, see Zhang and Shu (2006) and Luo et al. (2015).

4.3 Scalar Equation with Discontinuous Initial Solution

It is well known that the numerical oscillation will happen when the initial solution contains a discontinuous point and the piecewise polynomials $k \geq 1$. Consider the linear constant hyperbolic equation, namely $f(u) = \beta u$ in (1). The detailed analysis shows that the pollution region around the characteristic line across the discontinuous point is only restricted in a narrow zone, and the optimal error estimate outside the pollution region is also preserved. The next theorem (Zhang and Shu, 2014) stated this double-optimal result for RKDG(3,3,k) method with arbitrary $k \geq 0$.

Theorem 3. *Assume $T = M\tau$ for simplicity, and $k \geq 1$. Under the standard CFL condition, namely, $\tau \leq \lambda h$ with λ being suitably small, there is the optimal error estimate for RKDG(3,3,k) method to solve the linear constant hyperbolic equation*

$$\| u(\cdot, T) - u_h^M \|_{L^2(\mathbb{R} \setminus \mathcal{R}_T)} \leq C_1 (h^{k+1} + \tau^3),$$

out of the pollution region

$$\mathcal{R}_T = \left(\beta T - C_2 \sqrt{T\beta} h^{1/2} \log \frac{1}{h}, \beta T + C_3 \sqrt{T\beta} h^{1/2} \log \frac{1}{h} \right).$$

The above bounding constants are all independent of h, τ and λ^{-1}.

This theorem is proved by energy analysis with two special weight functions, in order to detect the left side and right side of pollution region. The analysis is long and complex, which involves many technical points, for example, the superconvergence results, the generalized slope function and the highest frequency component, as well as the complex treatment of those errors coming from the Runge–Kutta time marching.

The above result is independent on λ^{-1} and hence also holds for the semi-discrete DG method. The numerical results given in Zhang and Shu (2014) verify the sharpness of the above results.

Remark 4. Similar works have been carried out by many authors. For example, Johnson and Pitkäranta (1986) considered the space-time DG method and proved that the pollution region at the crosswind direction has the width of $\mathcal{O}(h^{1/2}\log(1/h))$, and Cockburn and Guzman (2008) considered RKDG (2,2,1) method and proved similar results.

4.4 Other Error Estimates

There are three kinds of studies on the superconvergence of DG methods. The first kind of study is that between the numerical solution and a special projection of exact solution, such as

$$\| \mathcal{P}_h u - u_h \| \leq C h^{k+1+\gamma},$$

where $\mathcal{P}_h u$ is the local Gauss–Radau projection, which is defined as an example, by

$$(\mathcal{P}_h u - u, v_h)_{I_i} = 0,\ \forall v_h \in \mathbb{P}^{k-1}(I_i);\ \ (\mathcal{P}_h u - u)^{-}_{i+\frac{1}{2}} = 0,\ \ i = 1,2,\ldots,N, \qquad (37)$$

for the linear constant hyperbolic with the flowing is from left to right. Here $\gamma > 0$ is the so-called superconvergence order. It is proved to be $\gamma = 1/2$ by Cheng and Shu (2010), and then it was developed to $\gamma = 1$ by Yang and Shu (2012). A nonlinear conservation law has been considered by Meng et al. (2012). The second one is that between the numerical solution and the exact solution on some special point in each element. For example, there hold $(k + 2)$-order accuracy at the Radau points and moreover $(2k + 1)$-order accuracy at the downwind endpoint. Detailed statement and technical proof can be found in Cao et al. (2014), where the modification polynomials based on Legendre polynomials play a very important role. The last one is the postprocessing of numerical solutions, such that

$$\| G_h * u_h - u \| \leq C h^{2k+1},$$

where G_h is a suitable kernel in the convolution manipulation. The main technique is the negative-norm analysis and the control on the difference quotient of the solution on the staggered mesh. For more details, refer to Cockburn et al. (2003) and Mirzaee et al. (2012) and references included therein.

The studies on the posterior estimate are also carried for a long time, although there are a few development, compared with the other error estimates. For example, Giles and Süli (2002) and Adjerid and Baccouch (2009) used the dual technique and construct an estimation indicator. It is worthy to point out that the superconvergence results can play very important role in this field.

5 LIMITERS FOR DISCONTINUOUS GALERKIN METHODS

An important component of RKDG methods for solving conservation laws, with strong shocks in the solutions, is a nonlinear limiter, which is applied

to detect discontinuities and control spurious oscillations near such discontinuities. There are many limiters which exist in the literature, for example, the minmod-type limiters, the moment-based limiters and the improved moment-based limiters. In this section, we will review these limiters and describe a robust limiter, the WENO-type limiter, which was developed in recent years.

Below we would like to use the notations in Cockburn and Shu (1989) to describe this procedure; however, we emphasize that the procedure actually does not depend on the specific basis chosen for the polynomials. In order to implement the DG methods for computation, we adopt a local orthogonal basis over I_i, $\{v_l^{(i)}(x),\ l=0,1,\ldots,k\}$, namely the scaled Legendre polynomials:

$$v_0^{(i)}(x)=1,\quad v_1^{(i)}(x)=\frac{x-x_i}{h_i/2},\quad v_2^{(i)}(x)=\left(\frac{x-x_i}{h_i/2}\right)^2-\frac{1}{3},\ldots,$$

where the points x_i are the centres of the cells $I_i=[x_{i-1/2},x_{i+1/2}]$. Then the numerical solution $u_h(x,t)$ in the space V_h^k can be written as

$$u_h(x,t)=\sum_{l=0}^{k}u_i^{(l)}(t)v_l^{(i)}(x),\ \text{for } x\in I_i \tag{38}$$

and the degrees of freedom $u_i^{(l)}(t)$ are the moments defined by

$$u_i^{(l)}(t)=\frac{1}{a_l}\int_{I_i}u_h(x,t)v_l^{(i)}(x)dx,\quad l=0,1,\ldots,k,$$

where $a_l=\int_{I_i}(v_l^{(i)}(x))^2dx$ are the normalization constants since the basis is not orthonormal. In order to determine the approximate solution, we evolve the degrees of freedom $u_i^{(l)}$:

$$\begin{aligned}\frac{d}{dt}u_i^{(l)}+\frac{1}{a_l}\Bigg(&-\int_{I_i}f(u_h(x,t))\frac{d}{dx}v_l^{(i)}(x)dx+\hat{f}(u_{i+1/2}^-,u_{i+1/2}^+)v_l^{(i)}(x_{i+1/2})\\&-\hat{f}(u_{i-1/2}^-,u_{i-1/2}^+)v_l^{(i)}(x_{i-1/2})\Bigg)=0,\quad l=0,1,\ldots,k,\end{aligned} \tag{39}$$

where $u_{i+1/2}^{\pm}=u_h(x_{i+1/2}^{\pm},t)$ are the left and right limits of the discontinuous solution u_h at the cell interface $x_{i+1/2}$, and $\hat{f}(u^-,u^+)$ is a monotone flux (nondecreasing in the first argument and nonincreasing in the second argument) for the scalar case and an exact or approximate Riemann solver for the system case. The integral term in (39) can be computed either exactly or by a suitable numerical quadrature accurate to at least $O(h^{k+l+2})$.

The semidiscrete scheme (39), written as $U_t=L(U)$, is then discretized in time by a nonlinearly stable Runge–Kutta time discretization which is described in Section 2.2. A limiter will be employed after each stage of time marching. For convenience of notations, below we would like to omit the time variable.

5.1 Traditional Limiters

Now we list a few traditional limiters, such as TVB limiter, moment limiter and modified moment limiter.

- The minmod-based TVB limiter (Cockburn and Shu, 1989). Denote:

$$u_{i+1/2}^{-}=u_i^{(0)}+\tilde{u}_i,\quad u_{i-1/2}^{+}=u_i^{(0)}-\tilde{\tilde{u}}_i.$$

From (38), we can see that

$$\tilde{u}_i=\sum_{l=1}^{k}u_i^{(l)}v_l^{(i)}(x_{i+1/2}),\quad \tilde{\tilde{u}}_i=-\sum_{l=1}^{k}u_i^{(l)}v_l^{(i)}(x_{i-1/2}).$$

These are modified by either the standard minmod limiter (Harten, 1983)

$$\tilde{u}_i^{(mod)}=m(\tilde{u}_i,\Delta_+u_i^{(0)},\Delta_-u_i^{(0)}),\quad \tilde{\tilde{u}}_i^{(mod)}=m(\tilde{\tilde{u}}_i,\Delta_+u_i^{(0)},\Delta_-u_i^{(0)}),\tag{40}$$

where $\Delta_+u_i^{(0)}=u_{i+1}^{(0)}-u_i^{(0)}$, $\Delta_-u_i^{(0)}=u_i^{(0)}-u_{i-1}^{(0)}$, and the minmod function m is given by

$$m(a_1,a_2,\ldots,a_n)=\begin{cases}s\cdot\min_{1\le j\le n}|a_j| & \text{if } \operatorname{sign}(a_1)=\operatorname{sign}(a_2)=\cdots=\operatorname{sign}(a_n)=s,\\ 0 & \text{otherwise,}\end{cases}\tag{41}$$

or the TVB modified minmod function (Shu, 1987)

$$\tilde{m}(a_1,a_2,\ldots,a_n)=\begin{cases}a_1 & \text{if } |a_1|\le Mh^2,\\ m(a_1,a_2,\ldots,a_n) & \text{otherwise,}\end{cases}\tag{42}$$

where $M>0$ is a constant. Then we reconstruct the new moment from $\tilde{u}_i^{(mod)}$, $\tilde{\tilde{u}}_i^{(mod)}$ and $u_i^{(0)}$. The choice of M depends on the solution of the problem. For scalar problems, it is possible to estimate M by the initial condition as in Cockburn and Shu (1989) (M is proportional to the second derivative of the initial condition at smooth extrema); however, it is more difficult to estimate M for the system case.

- Moment limiter of Biswas et al. (1994). The moment-based limiter in Biswas et al. (1994) is given by

$$u_i^{(l),mod}=\frac{1}{2l-1}\,m\Big((2l-1)u_i^{(l)},u_{i+1}^{(l-1)}-u_i^{(l-1)},u_i^{(l-1)}-u_{i-1}^{(l-1)}\Big),\tag{43}$$

where m is again the TVD minmod function (41). This limiter is applied adaptively. First, the highest-order moment $u^{(k)}$ is limited. Then the limiter is applied to successively lower-order moments when the next higher order moment on the interval has been changed by the limiting. In this way, the limiting is applied only where it is needed, and accuracy is retained in smooth regions.

- A modification of the moment limiter by Burbeau et al. (2001). If (43) is enacted, that is, $u_i^{(l),mod}\neq u_i^{(l)}$, then

$$\hat{u}_i^{(l),mod} = \frac{1}{2l-1} m\left((2l-1)u_i^{(l)}, u_{i+1/2}^{(l-1)+} - u_i^{(l-1)}, u_i^{(l-1)} - u_{i-1/2}^{(l-1)-}\right), \tag{44}$$

where

$$u_{i+1/2}^{(l-1)+} = u_{i+1}^{(l-1)} - (2l-1)u_{i+1}^{(l)}, \quad u_{i-1/2}^{(l-1)-} = u_{i-1}^{(l-1)} + (2l-1)u_{i-1}^{(l)}.$$

Again this limiter is applied adaptively as moment limiter.

These limiters tend to degrade accuracy when mistakenly used in smooth regions of the solution. In order to overcome the drawback of these limiters, from 2003, Qiu and colleagues have studied using WENO as limiter for RKDG methods, with the goal of obtaining a robust and high-order limiting procedure to simultaneously obtain uniform high-order accuracy and sharp, nonoscillatory shock transition for RKDG methods.

5.2 WENO Reconstruction as a Limiter for the RKDG Method

In this section, we will describe the procedure of WENO reconstruction as a limiter for the RKDG method (Qiu and Shu, 2005c; Zhu et al., 2008). The WENO-type limiter procedure is separated into two parts:

1. Identify the "troubled cells", namely those cells which might need the limiting procedure;
2. Reconstruct polynomials in "troubled cells" using WENO reconstruction which only maintain the original cell averages (conservation).

For the first part, there are many troubled cell indicators which can be used. In Qiu and Shu (2005a), we have systematically studied and compared a few different troubled cell indicators for the RKDG methods using WENO methodology as limiters. Extensive one- and two-dimensional simulations on the hyperbolic systems of Euler equations indicate that the minmod-based TVB indicator (when the TVB constant M is suitably chosen) and the KXRCF indicator by Krivodonova et al. (2004) are better than other choices in all the test cases. Recently, the troubled cell indicators based on wavelets and outlier detectors were presented by Vuik and Ryan (2014):

- TVB minmod troubled cell indicator, if, in (40), we have $\tilde{u}_i^{(mod)} \neq \tilde{u}_i$ or $\widetilde{\tilde{u}}_i^{(mod)} \neq \widetilde{\tilde{u}}_i$, then the cell is identified as a troubled cell, and be marked for further reconstruction.
- A shock detection technique based on a strong superconvergence at the outflow boundary of each element in smooth regions for the discontinuous Galerkin method by Krivodonova et al. (2004). We will denote the troubled cell indicator as the KXRCF indicator. The KXRCF indicator can be described as follows.

  Partition the boundary of a cell I_i into two portions ∂I_i^- (inflow, $\vec{v} \cdot \vec{n} < 0$) and ∂I_i^+ (outflow, $\vec{v} \cdot \vec{n} > 0$). The cell I_i is identified as a troubled cell marked for further reconstruction, if

$$\frac{\left|\int_{\partial I_i^-}(u^h|_{I_i}-u^h|_{I_{n_i}})ds\right|}{h_i^{\frac{k+1}{2}}\left|\partial I_i^-\right|||u^h|_{I_i}||}>1,$$

where h_i is the radius of the circumscribed circle in the element I_i. I_{n_i} is the neighbour of I_i on the side of ∂I_i^- and the norm is based on an element average in one-dimensional case.

Let I_i be a troubled cell which is identified by one of the troubled cell indicators which are described above, we will reconstruct the degrees of freedom, or the moments, $u_i^{(l)}$ for the troubled cell I_i for $l = 1, \ldots, k$ and retain only the cell average $u_i^{(0)}$.

- We identify k+1 small stencils $S_j, j = 0, \ldots, k$, such that I_i belongs to each of them. Here we set $S_j = \cup_{l=0}^{k} I_{i+j-l}$. We denote by $\mathcal{T} = \cup_{j=0}^{k} S_j$ the larger stencil which contains all the cells from the k+1 smaller stencils.

  We have a kth degree polynomial reconstruction denoted by $q_j(x)$, associated with each of the stencils $S_j, j = 0, \ldots, k$, such that the cell average of $q_j(x)$ in each of the cells in the stencil S_j agrees with the given cell average of u, i.e.

$$\frac{1}{\Delta x_{i+j-l}}\int_{I_{i+j-l}} q_j(x)dx = u_{i+j-l}^{(0)}, l = 0, \ldots, k.$$

We also have a higher order $(2k)$th degree polynomial reconstruction denoted by $Q(x)$, associated with the larger stencil $\mathcal{T}$, such that

$$\frac{1}{\Delta x_{i+l}}\int_{I_{i+l}} Q(x)dx = u_{i+l}^{(0)}, l = -k, \ldots, k.$$

The detail of the construction of $q_j(x)$ and $Q(x)$ can be found in Shu (1998).

- We find the combination coefficients, also called linear weights $\gamma_j, j = 0, 1, \ldots, k$ satisfying:

$$A: \int_{I_i} Q(x)v_l^{(i)}(x)dx = \sum_{j=0}^{k}\gamma_j\int_{I_i} q_j(x)v_l^{(i)}(x)dx, \quad l = 1, \ldots, k;$$

$$B: Q(x_G) = \sum_{j=0}^{k}\gamma_j q_j(x_G).$$

- We compute the smoothness indicator, denoted as β_j for each stencil S_j, which measures how smooth the function $q_j(x)$ on cell I_i,

$$\beta_j = \sum_{l=1}^{k}\int_{x_{i-1/2}}^{x_{i+1/2}} (\Delta x)^{2l-1}(q_j^{(l)})^2 dx,$$

  where $q_j^{(l)}$ is the lth derivative of $q_j(x)$.

- We compute the nonlinear weight ω_j based on the smoothness indicator

$$\omega_j = \frac{\alpha_j}{\sum_{l=0}^{k} \alpha_l}, \text{ with } \alpha_j = \frac{\gamma_j}{(\varepsilon + \beta_j)^2}, j = 0, 1, \ldots, k,$$

  where $\varepsilon > 0$ is a small number to avoid the denominator to become 0.
- The final WENO approximation is then given by:

$$A : u_i^{(l)} = \frac{1}{a_l} \sum_{j=0}^{k} \omega_j \int_{I_i} q_j(x) v_l^{(i)}(x) dx, \quad l = 1, \ldots, k;$$

$$B : u(x_G) = \sum_{j=0}^{k} \omega_j q_j(x_G).$$

- Reconstruction of moments based on the reconstructed point values for procedure B:

$$u_i^{(l)} = \frac{\Delta x}{a_l} \sum_{G} w_G u(x_G) v_l^{(i)}(x_G), \quad l = 1, \ldots, k.$$

Remark 5. For procedure *A*, there are not the linear weights for $\mathbb{P}^3$ case. For procedure *B*, we use the two-point and four-point Gauss quadrature in the $\mathbb{P}^1$ and $\mathbb{P}^3$ cases, respectively. For the $\mathbb{P}^2$ case, we use either the four-point Gauss–Lobatto quadrature points or three-point Gauss quadrature points. But there are negative linear weights when three-point Gauss quadrature points are used.

Remark 6. For the system cases, in order to achieve better qualities at the price of more complicated computations, the WENO reconstruction limiter is always used with a local characteristic field decomposition (see, e.g., Shu, 1998 for details).

5.3 Hermite WENO Reconstruction

WENO limiters work well in all our numerical test cases, including 1D, 2D and 3D, structure and unstructured meshes (Qiu and Shu, 2005c; Zhu et al., 2008; Zhu and Qiu, 2012), but for $\mathbb{P}^2$ and $\mathbb{P}^3$ cases, the compactness of DG is destroyed. In order to maintain the compactness of DG methods, we developed the following Hermite WENO (HWENO) limiter (Qiu and Shu, 2004, 2005b). For $\mathbb{P}^2$ case, we summarize the procedure to reconstruct the first and second moments $u_i^{(1)}$ and $u_i^{(2)}$ for a troubled cell I_i using HWENO. First, we reconstruct the following polynomials:

$$\int_{I_{i+j}} q_0(x) dx = u_{i+j}^{(0)} a_0, \quad j = -1, 0; \qquad \int_{I_{i-1}} q_0(x) v_1^{(i-1)}(x) dx = u_{i-1}^{(1)} a_1;$$

$$\int_{I_{i+j}} q_1(x) dx = u_{i+j}^{(0)} a_0, \quad j = 0, 1; \qquad \int_{I_{i+1}} q_1(x) v_1^{(i+1)}(x) dx = u_{i+1}^{(1)} a_1;$$

$$\int_{I_{i+j}} q_2(x)dx = u_{i+j}^{(0)} a_0, \quad j=-1,0,1;$$

$$\int_{I_{i+j}} Q(x)dx = u_{i+j}^{(0)} a_0, \quad j=-1,0,1; \qquad \int_{I_{i+j}} Q(x) v_1^{(i+j)}(x)dx = u_{i+j}^{(1)} a_1, \quad j=-1,1,$$

Then, following the routine A of WENO reconstruction, we can obtain the new moment $u_i^{(1)}$. To reconstruct $u_i^{(2)}$, we have the following polynomials:

$$\int_{I_{i+j}} q_0(x)dx = u_{i+j}^{(0)} a_0, \qquad \int_{I_{i+j}} q_0(x) v_1^{(i+j)}(x)dx = u_{i+j}^{(1)} a_1, \quad j=-1,0;$$

$$\int_{I_{i+j}} q_1(x)dx = u_{i+j}^{(0)} a_0, \qquad \int_{I_{i+j}} q_1(x) v_1^{(i+j)}(x)dx = u_{i+j}^{(1)} a_1, \quad j=0,1;$$

$$\int_{I_{i+j}} q_2(x)dx = u_{i+j}^{(0)} a_0, \quad j=-1,0,1; \qquad \int_{I_i} q_2(x) v_1^{(i)} dx = u_i^{(1)} a_1;$$

$$\int_{I_{i+j}} Q(x)dx = u_{i+j}^{(0)} a_0, \qquad \int_{I_{i+j}} Q(x) v_1^{(i+j)}(x)dx = u_{i+j}^{(1)} a_1, \quad j=-1,0,1.$$

Then following the routine A of WENO reconstruction, we can obtain the new moment $u_i^{(2)}$.

5.4 A Simple WENO-Type Limiter

In 2013, Zhong and Shu developed a simple WENO-type limiter for DG (Zhong and Shu, 2013). Let I_i be a troubled cell, we use stencil $S = \{I_{i-1}, I_i, I_{i+1}\}$. Denote the solutions of the DG method on these three cells as polynomials $q_0(x)$, $q_1(x)$ and $q_2(x)$, respectively. We would like to modify $q_1(x)$ to $q_1^{new}(x)$. In order to make sure that the reconstructed polynomial maintains the original cell average of q_1 in the target cell I_i, the following modifications are taken:

$$\widetilde{q}_0 = q_0 - \bar{\bar{q}}_0 + \bar{\bar{q}}_1, \quad \widetilde{q}_1 = q_1, \quad \widetilde{q}_2 = q_0 - \bar{\bar{q}}_2 + \bar{\bar{q}}_1,$$

$$\bar{\bar{q}}_0 = \frac{1}{\Delta x_i}\int_{I_i} q_0(x)dx, \quad \bar{\bar{q}}_1 = \frac{1}{\Delta x_i}\int_{I_i} q_1(x)dx, \quad \bar{\bar{q}}_2 = \frac{1}{\Delta x_i}\int_{I_i} q_2(x)dx.$$

The final nonlinear WENO reconstruction polynomial $q_1^{new}(x)$ is now defined by a convex combination of these modified polynomials:

$$q_1^{new}(x) = \omega_0 \widetilde{q}_0(x) + \omega_1 \widetilde{q}_1(x) + \omega_2 \widetilde{q}_2(x).$$

If $\omega_0 + \omega_1 + \omega_2 = 1$, then q_1^{new} has the same cell average and order of accuracy as q_1.

Computational formulae of ω_0, ω_1 and ω_2 are the same as in WENO reconstruction. The linear weights can be chosen to be any set of positive

numbers adding up to one. Since for smooth solutions the central cell is usually the best one, a larger linear weight is put on the central cell than on the neighbouring cells, i.e.

$$\gamma_0 < \gamma_1 \quad \text{and} \quad \gamma_1 > \gamma_2.$$

In Zhong and Shu (2013), they take:

$$\gamma_0 = 0.001, \quad \gamma_1 = 0.998, \quad \gamma_2 = 0.001,$$

which can maintain the original high order in smooth regions and can keep essentially nonoscillatory shock transitions in all their numerical examples.

5.5 A Simple and Compact HWENO Limiter

This new HWENO limiter (Zhu et al., 2016) is an modification of the simple WENO limiter proposed recently by Zhong and Shu (2013). Both limiters use information of the DG solutions only from the target cell and its immediate neighbouring cells, thus maintaining the original compactness of the DG scheme. The goal of both limiters is to obtain high-order accuracy and nonoscillatory properties simultaneously. The main novelty of the new HWENO limiter in this chapter is to reconstruct the polynomial on the target cell in a least square fashion, while the simple WENO limiter (Zhong and Shu, 2013) is to use the entire polynomial of the original DG solutions in the neighbouring cells with an addition of a constant for conservation. The modification improves the robustness in the computation of problems with strong shocks or contact discontinuities and can get better resolutions for some examples for the P^3 case without the help of positivity-preserving limiters, and without changing the compact stencil of the DG scheme.

In order to make sure that the reconstructed polynomial maintains the original cell average of q_1 in the troubled cell I_i, the following modifications are taken:

$$\int_{I_{i-1}} (\widetilde{q}_0(x) - q_0(x))^2 dx = \min \int_{I_{i-1}} (\phi(x) - q_0(x))^2 dx,$$

$$\int_{I_{i+1}} (\widetilde{q}_2(x) - q_2(x))^2 dx = \min \int_{I_{i+1}} (\phi(x) - q_2(x))^2 dx$$

for $\forall \phi(x) \in \mathbb{P}^k$ with $\int_{I_i} \phi(x)dx = \int_{I_i} q_1(x)dx$.

For notational consistency, we also denote $\widetilde{q}_1(x) = q_1(x)$. Then we follow the routine of Zhong and Shu (2013) and obtain the final nonlinear WENO reconstruction polynomial $q_1^{new}(x)$.

For two-dimensional cases, the reconstruction procedure of the limiters refers to Cockburn et al. (1990), Cockburn and Shu (1998), Biswas et al. (1994), Burbeau et al. (2001), Qiu and Shu (2005c), Qiu and Shu (2005b),

Zhu et al. (2008), Zhu and Qiu (2009), Zhu and Qiu (2011), Zhu and Qiu (2012), Zhong and Shu (2013), Zhu et al. (2013), and Zhu et al. (2016, submitted for publication).

6 CONCLUDING AND REMARKS

In this chapter, we reviewed the stability, error estimates and limiters for RKDG methods. For the stability, we focused on nonlinear stability by the energy analysis strategy; for error estimates, the priori estimate and posterior estimate are reviewed, and the traditional limiters and WENO-type limiters are shown in Section 5.

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